Survival probabilities in a discrete semi-Markov risk model

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In this paper, we consider the survival probability for a discrete semi-Markov risk model, which assumes individual claims are influenced by a Markov chain with finite state space and there is autocorrelation among consecutive claim sizes. Our semi-Markov risk model is similar to the one studied in Reinhard and Snoussi [1, 2] without the restriction imposed on the distributions of the claims. In particular, the model of study includes several existing risk models such as the compound binomial model (with time-correlated claims) and the compound Markov binomial model (with time-correlated claims) as special cases. The main purpose of the paper is to develop a recursive method for computing the survival probability in the two-state model, and present some numerical examples to illustrate the application of our results.

Keywords: generating function; recursive formula; semi-Markov risk model; survival probability

1. Introduction

Markov-modulated risk models, where the surplus processes are influenced by an environmental Markov chain, has attracted a lot of attention recently. Some recent papers on ruin problems for these models include Lu and Li [3, 4], Lu [5], Ng and Yang [6], Zhu and Yang [7, 8], Zhang [9], Diko and Usábel [10] and references therein.

In a Markov-modulated risk model, the premiums, claim amounts and claim number process are usually assumed to be (conditionally) independent given the environmental Markov chain, that is, they only depend on the current state of the Markov chain. However, this (conditional) independence assumption may be somewhat too strong in some applications. Janssen and Reinhard [11] first considered a semi-Markovian dependence structure where the claim amounts and inter-claim times not only depend on the current state but also the next state of the environmental Markov chain. They derived the survival probabilities in terms of an infinite series of matrix convolutions. Albrecher and Boxma [12] considerably generalized the approach of Janssen and Reinhard [11] and investigated the discounted penalty function in such a risk model by means of Laplace-Stieltjes transforms. Recently, Cheung and Landriault [13] further studied the work of Albrecher and Boxma [12] by relaxing some assumptions pertaining to the inter-claim time distribution.

With a strict restriction imposed on the total claim amount, Reinhard and Snoussi [1, 2] considered a discrete-time semi-Markov risk model and derived recursive formulae for

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calculating the distribution of the surplus just prior to ruin and the distribution of the deficit at ruin in a special case. In this paper, we shall relax the restriction of Reinhard and Snoussi [1, 2] and derive closed-form expressions for the ruin probability in the two-state semi-Markov risk model. Since the model of study embraces some existing discrete-time risk models including the compound binomial model (with time-correlated claims) and the compound Markov binomial model (with time-correlated claims), the present paper generalizes the study of ruin probability for these risk models.

The rest of the paper is organized as follows. In Section 2, we present the mathematical formulation of the discrete semi-Markov model. In Section 3, we derive recursive formulae for computing survival probabilities for the model. Section 4 is devoted to finding the initial values for applying the recursive formulae. Several special cases of our model are considered in Section 5. Finally, some numerical examples are presented in Section 6.

2. The model

The model considered in this paper is based on a discrete-time semi-Markov risk model proposed by Reinhard and Snoussi [1, 2]. Let $(J_n, n \in \mathbb{N})$ be a homogeneous, irreducible and aperiodic Markov chain with finite state space $M = \{1, \cdots, m\}$ $(1 \le m < \infty)$. Its one-step transition probability matrix is given by

$$\mathbf{P} = (p_{ij})_{i,j \,\in M}, \qquad p_{ij} = \mathbb{P}(J_n = j | J_{n-1} = i, J_k, k \le n-1),$$

with a unique stationary distribution $\boldsymbol{\pi} = (\pi_1, \cdots, \pi_m)$. The insurer's surplus at the end of the t-th period ($t \in \mathbb{N}_+$), U_t, has the form

$$U_t = u + ct - \sum_{i=1}^{t} Y_i, \qquad t \in \mathbb{N}_+, \tag{1}$$

where Y_i denotes the total claim amount in the i-th period, and $c \in \mathbb{N}_+ = \{1, 2, \cdots\}$ is the amount of premium per period. We further assume that the insurer has a non-negative initial surplus u, and that Y_t's are nonnegative integer-valued random variables. The distribution of Y_t's is influenced by the environmental Markov chain $(J_n, n \in \mathbb{N})$ in the way that (J_t, Y_t) depends on $\{J_k, Y_k; k \le t-1\}$ only through J_{t-1}. Define

$$g_{ij}(l) = \mathbb{P}(Y_t = l, J_t = j | J_{t-1} = i, J_k, Y_k, k \le t-1), \quad l \in \mathbb{N},$$

which describes the conditional joint distribution of Y_t and J_t given the previous state J_{t-1}, and plays a key role in the following derivations. Note that the variables $\{Y_t, t \in \mathbb{N}_+\}$ are conditionally independent given the environmental Markov chain.

Assume that premiums are received at the beginning of each time period with $c = 1$. As was mentioned in Gerber et al. [14], this is the case when claim amounts are multiples of the periodic premium; and hence, after a change of monetary units, the periodic premium is 1. Assume further that for all i and j,

$$\mu_{ij} = \sum_{k=0}^{\infty} k g_{ij}(k) < \infty,$$

and define

$$\mu_i = \sum_{j=1}^{m} \mu_{ij},\ i \in M.$$

Define $\tau = \inf\{t \in \mathbb{N}_+ : U_t < 0\}$ as the time of ruin and let

$$\psi_i(u) = \mathbb{P}(\tau < \infty | U_0 = u, J_0 = i), \quad i \in M,\ u \in \mathbb{N}$$

be the ultimate ruin probability given the initial surplus u and the initial environment state i. Let $\phi_i(u) = 1 - \psi_i(u)$ be the corresponding survival probability. To make sure that ruin is not certain, we assume that the positive safety loading condition holds, that is, $\sum_{i=1}^{m} \pi_i \mu_i < 1$. Here our aim is to derive recursive formulae for computing $\phi_i(u)$.

Remark 1. *In Reinhard and Snoussi [1, 2], it is assumed that*

$$\begin{cases} g_{ij}(0) = 0, \quad \forall j \in M, \quad i \neq 1, \\ \sum_{j \in M} g_{1j}(0) > 0. \end{cases} \tag{2}$$

This condition says that zero claims are only possible when the state prior to the occurrence of the claim is state 1. Without this condition, the recursive formulae cannot be established using their method even for $m = 2$. Here we relax the condition, and consider the case of $m = 2$. Furthermore, without restriction (2), model (1) covers the compound binomial model (with time-correlated claims) and the compound Markov binomial model (with time-correlated claims) as special cases. For details, see Section 5.

3. Recursive formulae

In this section, we derive recursive formulae for computing survival probabilities $\phi_i(u), i = 1, 2$. To do this, we adopt the method of Chen et al. [15] in which a dividend problem for the same discrete semi-Markov risk model was considered.

Considering the first time period, we obtain the following recursion

$$\phi_i(u) = \sum_{j=1}^{2} \sum_{k=0}^{u+1} g_{ij}(k) \phi_j(u + 1 - k), \quad i = 1, 2,\ u \in \mathbb{N}. \tag{3}$$

In order to obtain recursive formulae for computing $\phi_i(u)$, we employ the technique of generating functions. Let $\widetilde{\phi}_i(s)$ and $\widetilde{g}_{ij}(s)$ denote the generating functions of $\phi_i(k)$ and $g_{ij}(k)$ respectively. By multiplying both sides of (3) by s^{u+1} and summing over u from 0 to ∞, we obtain

$$s\widetilde{\phi}_i(s) = \sum_{j=1}^{2} \widetilde{g}_{ij}(s) \widetilde{\phi}_j(s) - \sum_{j=1}^{2} g_{ij}(0) \phi_j(0), \qquad i = 1, 2.$$

Let $e_i = \sum_{j=1}^{2} g_{ij}(0) \phi_j(0),\ i = 1, 2$. Then we have

$$\begin{cases} (\widetilde{g}_{11}(s) - s)\widetilde{\phi}_1(s) + \widetilde{g}_{12}(s)\widetilde{\phi}_2(s) = e_1, \\ \widetilde{g}_{21}(s)\widetilde{\phi}_1(s) + (\widetilde{g}_{22}(s) - s)\widetilde{\phi}_2(s) = e_2. \end{cases} \tag{4}$$

It follows from (4) that

$$[(\widetilde{g}_{11}(s)-s)(\widetilde{g}_{22}(s)-s)-\widetilde{g}_{21}(s)\widetilde{g}_{12}(s)]\,\widetilde{\phi}_1(s)=e_1(\widetilde{g}_{22}(s)-s)-e_2\widetilde{g}_{12}(s). \tag{5}$$

For notational convenience, we define

$$\begin{aligned}
&\bar{g}_{ii}(1)=g_{ii}(1)-1,\quad \bar{g}_{ii}(k)=g_{ii}(k),\quad i=1,2,\quad k\in\mathbb{N}\backslash\{1\};\\
&f_k=\sum_{n=0}^{k}\Big[\bar{g}_{11}(n)\bar{g}_{22}(k-n)-g_{21}(n)g_{12}(k-n)\Big],\\
&g_k^{(1)}=\sum_{n=0}^{k}\phi_1(n)f_{k-n},\quad h_k^{(1)}=e_1\bar{g}_{22}(k)-e_2g_{12}(k),\quad k\in\mathbb{N}.
\end{aligned}$$

Let $\widetilde{g^{(1)}}(s)$, $\widetilde{f}(s)$ and $\widetilde{h^{(1)}}(s)$ denote the generating functions of $g_k^{(1)}$, $f(k)$ and $h_k^{(1)}$ respectively. Note that for any two sequences $\{a(n),n=0,1,\cdots\}$ and $\{b(n),n=0,1,\cdots\}$ with generating functions $\widetilde{a}(s)$ and $\widetilde{b}(s)$, we have the following property

$$\widetilde{a}(s)\widetilde{b}(s)=\sum_{n=0}^{\infty}a*b(n)s^n=\sum_{n=0}^{\infty}\sum_{k=0}^{n}a(k)b(n-k)s^n.$$

Applying this property to (5) yields

$$\widetilde{g^{(1)}}(s)=\widetilde{f}(s)\widetilde{\phi}_1(s)=\widetilde{h^{(1)}}(s).$$

Then comparing the coefficients of s^k in both sides of the above equation gives $g_k^{(1)}=h_k^{(1)},k\in\mathbb{N}$, that is,

$$\sum_{n=0}^{k}\phi_1(n)f_{k-n}=h_k^{(1)},\qquad k\in\mathbb{N}. \tag{6}$$

Similarly, one can obtain

$$\sum_{n=0}^{k}\phi_2(n)f_{k-n}=h_k^{(2)},\qquad k\in\mathbb{N}, \tag{7}$$

where $h_k^{(2)}=-e_1g_{21}(k)+e_2\bar{g}_{11}(k)$.

Proposition 1. *If both $f_0=0$ and $f_1=0$, then $\pi_1\mu_1+\pi_2\mu_2\geq 1$, that is, the positive safety loading condition does not hold.*

Proof. It is easy to see that the unique stationary distribution is

$$\boldsymbol{\pi}=(\pi_1,\pi_2)=\Big(\frac{p_{21}}{p_{21}+p_{12}},\frac{p_{12}}{p_{21}+p_{12}}\Big). \tag{8}$$

On the other hand,

$$f_1=g_{11}(0)\left[g_{22}(1)-1\right]+g_{22}(0)\left[g_{11}(1)-1\right]-g_{21}(0)g_{12}(1)-g_{21}(1)g_{12}(0)\leq 0.$$

So if $f_1=0$, then

$$g_{11}(0)=0,\quad g_{22}(0)=0,\quad g_{21}(0)g_{12}(1)=0,\quad g_{21}(1)g_{12}(0)=0. \tag{9}$$

In addition, $g_{21}(0)g_{12}(0)=0$ since $f_0=0$. Hence there are only two possibilities:

(i) $g_{12}(0) = 0$ and $g_{21}(0) \neq 0$: From (9), we see $g_{12}(1) = 0$. Then

$$\mu_1 = \sum_{k=0}^{\infty} k[g_{11}(k) + g_{12}(k)] = \sum_{k=1}^{\infty} k g_{11}(k) + \sum_{k=2}^{\infty} k g_{12}(k) \geq p_{11} + 2p_{12} = 1 + p_{12},$$

$$\mu_2 = \sum_{k=0}^{\infty} k[g_{21}(k) + g_{22}(k)] \geq p_{21} + p_{22} - g_{21}(0) = 1 - g_{21}(0).$$

Therefore,

$$\pi_1\mu_1 + \pi_2\mu_2 = \frac{p_{21}\mu_1 + p_{12}\mu_2}{p_{21} + p_{12}} \geq \frac{p_{21} + p_{12} + p_{12}(p_{21} - g_{21}(0))}{p_{21} + p_{12}} \geq 1.$$

(ii) $g_{21}(0) = 0$ and $g_{12}(0) \neq 0$: By similar arguments, we have

$$g_{21}(1) = 0, \quad \mu_1 \geq 1 - g_{12}(0), \quad \mu_2 \geq 1 + p_{21},$$

and

$$\pi_1\mu_1 + \pi_2\mu_2 = \frac{p_{21}\mu_1 + p_{12}\mu_2}{p_{21} + p_{12}} \geq \frac{p_{21} + p_{12} + p_{21}(p_{12} - g_{12}(0))}{p_{21} + p_{12}} \geq 1.$$

□

Finally, from (6), (7) and Proposition 1, we obtain the following recursive formulae

$$\phi_i(k) = \begin{cases} \frac{1}{f_0}\left[h_k^{(i)} - \sum_{n=0}^{k-1} \phi_i(n) f_{k-n}\right] & \text{if } f_0 \neq 0, \\ \frac{1}{f_1}\left[h_{k+1}^{(i)} - \sum_{n=0}^{k-1} \phi_i(n) f_{k+1-n}\right] & \text{if } f_0 = 0 \text{ and } f_1 \neq 0. \end{cases} \tag{10}$$

for $i = 1, 2$ and $k \in \mathbb{N}_+$.

4. The initial value for $\phi_i(u)$

After obtaining (10) for computing $\phi_i(u)$, we need to determine the initial values $\phi_i(0)$, $i = 1, 2$. In Chen et al. [15], they took full advantage of the boundary conditions for barrier dividend strategy to obtain the initial values for the expected discounted dividends. Unfortunately, their method is inapplicable here. In order to calculate $\phi_1(0)$ and $\phi_2(0)$, we need to find two equations associated with them.

4.1. The first equation

For $i = 1, 2$, define

$$d_i(0) = \phi_i(0), \qquad d_i(u) = \phi_i(u) - \phi_i(u-1), \quad u \geq 1,$$
$$A_i(0) = h_0^{(i)}, \qquad A_i(u) = h_u^{(i)} - h_{u-1}^{(i)}, \quad u \geq 1.$$

It follows from (6) and (7) that

$$
\begin{aligned}
f_0 d_i(u+1) &= A_i(u+1) - \sum_{n=0}^{u} \phi_i(n) f_{u+1-n} - \sum_{n=0}^{u-1} \phi_i(n) f_{u-n} \\
&= A_i(u+1) - d_i(0) f_{u+1} - \sum_{n=1}^{u} \phi_i(n) f_{u+1-n} + \sum_{n=0}^{u-1} \phi_i(n) f_{u-n} \\
&= A_i(u+1) - d_i(0) f_{u+1} - \sum_{n=1}^{u} d_i(n) f_{u+1-n},
\end{aligned}
$$

for $u \in \mathbb{N}$. That is, for $u \in \mathbb{N}$,

$$
\sum_{l=0}^{u+1} f_l d_i(u+1-l) = A_i(u+1). \tag{11}
$$

Let $\widetilde{f}(s)$, $\widetilde{d}_i(s)$ and $\widetilde{A}_i(s)$ denote the generating functions of $f_u, d_i(u)$ and $A_i(u)$ respectively. Then, by (11), we have

$$
\widetilde{f}(s)\widetilde{d}_i(s) - f_0 d_i(0) = \widetilde{A}_i(s) - A_i(0), i = 1, 2.
$$

Since $A_i(0) = f_0 d_i(0)$, we obtain

$$
\widetilde{f}(s)\widetilde{d}_i(s) = \widetilde{A}_i(s), \quad i = 1, 2, \tag{12}
$$

which in turn yield

$$
\widetilde{A}_i'(s) = \widetilde{f}'(s)\widetilde{d}_i(s) + \widetilde{f}(s)\widetilde{d}_i'(s), \quad i = 1, 2. \tag{13}
$$

Since

$$
\widetilde{d}_i(1) = \sum_{u=0}^{\infty} d_i(u) = \lim_{n\to\infty} \phi_i(n) = 1, \quad \text{and} \quad \widetilde{A}_i(1) = \lim_{n\to\infty} A_i(n) = 0,
$$

it follows from (12) that $\widetilde{f}(1) = 0$ which in turn gives $\widetilde{A}_i'(1) = \widetilde{f}'(1)$ due to (13).

Note that $h_k^{(1)}$ can be rewritten as

$$
h_k^{(1)} = \xi_k^{(1)} \phi_1(0) + \eta_k^{(1)} \phi_2(0), \qquad k \in \mathbb{N}, \tag{14}
$$

where

$$
\xi_k^{(1)} = g_{11}(0)\bar{g}_{22}(k) - g_{21}(0) g_{12}(k), \quad \text{and} \quad \eta_k^{(1)} = g_{12}(0)\bar{g}_{22}(k) - g_{22}(0) g_{12}(k).
$$

So we have

$$
\begin{aligned}
\widetilde{A}_1'(1) &= \sum_{k=1}^{\infty} k(h_k^{(1)} - h_{k-1}^{(1)}) = \sum_{k=1}^{\infty}\sum_{i=1}^{k} (h_k^{(1)} - h_{k-1}^{(1)}) = \sum_{i=1}^{\infty}\sum_{k=i}^{\infty} (h_k^{(1)} - h_{k-1}^{(1)}) \\
&= \sum_{i=1}^{\infty} (-h_{i-1}^{(1)} + \lim_{k\to\infty} h_k^{(1)}) = -\sum_{i=0}^{\infty} h_i^{(1)} = -\phi_1(0) \sum_{i=0}^{\infty} \xi_i^{(1)} - \phi_2(0) \sum_{i=0}^{\infty} \eta_i^{(1)} \\
&= \phi_1(0)\Big(g_{11}(0) p_{21} + g_{21}(0) p_{12}\Big) + \phi_2(0)\Big(g_{12}(0) p_{21} + g_{22}(0) p_{12}\Big).
\end{aligned}
$$

On the other hand,

$$\begin{aligned}
\widetilde{f}'(1) &= \sum_{k=0}^{\infty} k f_k = \sum_{k=0}^{\infty}\sum_{n=0}^{k} k[\bar{g}_{11}(n)\bar{g}_{22}(k-n) - g_{21}(n)g_{12}(k-n)] \\
&= \sum_{n=0}^{\infty}\sum_{k=n}^{\infty} k[\bar{g}_{11}(n)\bar{g}_{22}(k-n) - g_{21}(n)g_{12}(k-n)] \\
&= \sum_{n=0}^{\infty}\Big[\bar{g}_{11}(n)\sum_{k=n}^{\infty}(k-n)\bar{g}_{22}(k-n) + n\bar{g}_{11}(n)\sum_{k=n}^{\infty}\bar{g}_{22}(k-n)\Big] \\
&\qquad - \sum_{n=0}^{\infty}\Big[g_{21}(n)\sum_{k=n}^{\infty}(k-n)g_{12}(k-n) + ng_{21}(n)\sum_{k=n}^{\infty}g_{12}(k-n)\Big] \\
&= \sum_{n=0}^{\infty}\bar{g}_{11}(n)\sum_{k=0}^{\infty}k\bar{g}_{22}(k) + \sum_{n=0}^{\infty}n\bar{g}_{11}(n)\sum_{k=0}^{\infty}\bar{g}_{22}(k) \\
&\qquad - \sum_{n=0}^{\infty}g_{21}(n)\sum_{k=0}^{\infty}kg_{12}(k) - \sum_{n=0}^{\infty}ng_{21}(n)\sum_{k=0}^{\infty}g_{12}(k) \\
&= -p_{12}(\mu_{22}-1) - p_{21}(\mu_{11}-1) - p_{21}\mu_{12} - p_{12}\mu_{21} \\
&= p_{12}(1-\mu_2) + p_{21}(1-\mu_1).
\end{aligned}$$

As a result, we obtain

$$\begin{aligned}
&\phi_1(0)\Big(g_{11}(0)p_{21} + g_{21}(0)p_{12}\Big) + \phi_2(0)\Big(g_{12}(0)p_{21} + g_{22}(0)p_{12}\Big) \\
=\ & p_{12}(1-\mu_2) + p_{21}(1-\mu_1). \qquad (15)
\end{aligned}$$

Remark 2. *By (8) and the positive safety loading condition $\pi_1\mu_1 + \pi_2\mu_2 < 1$, it is easy to see that $p_{12}(1-\mu_2) + p_{21}(1-\mu_1) > 0$. Along the same lines, one can derive the same expression for $\widetilde{A}_2'(1)$, that is, the two expressions for $\widetilde{A}_1'(1)$ and $\widetilde{A}_2'(1)$ (derived using the same method) coincide. Hence we need to seek an alternative method for constructing the second equation.*

4.2. The second equation

In this subsection, we use an alternative method to find another relation between $\phi_1(0)$ and $\phi_2(0)$. To do it, we consider several cases of f_0.

Case 1. If $f_0 = 0$, it follows from (6) that

$$f_1\phi_1(0) = h_1^{(1)} = \xi_1^{(1)}\phi_1(0) + \eta_1^{(1)}\phi_2(0),$$

which yields

$$K_1\phi_1(0) + K_2\phi_2(0) = 0, \qquad (16)$$

where

$$\begin{aligned}
K_1 &= \xi_1^{(1)} - f_1 = g_{12}(0)g_{21}(1) + g_{22}(0)(1 - g_{11}(1)), \\
K_2 &= \eta_1^{(1)} = g_{12}(0)(g_{22}(1) - 1) - g_{22}(0)g_{12}(1).
\end{aligned}$$

Furthermore,

$$K_1 = K_2 = 0 \iff g_{12}(0) = g_{22}(0) = 0.$$

In this case, we have $e_1 = g_{11}(0)\phi_1(0)$, $e_2 = g_{21}(0)\phi_1(0)$, and

$$f_1\phi_2(0) = h_1^{(2)} = [-g_{11}(0)g_{21}(1) + g_{21}(0)(g_{11}(1) - 1)]\phi_1(0). \tag{17}$$

Case 2. Suppose that $f_0 > 0$. Let

$$H_1(s) = (\widetilde{g}_{11}(s) - s)(\widetilde{g}_{22}(s) - s) - \widetilde{g}_{21}(s)\widetilde{g}_{12}(s),$$
$$H_2(s) = e_1(\widetilde{g}_{22}(s) - s) - e_2\widetilde{g}_{12}(s).$$

By (5), we have $H_1(s)\widetilde{\phi}_1(s) = H_2(s)$. Since

$$H_2(0) = f_0\phi_1(0) > 0, \quad \text{and} \quad H_2(1) = e_1(p_{22} - 1) - e_2p_{12} < 0,$$

there exists a $\rho \in (0, 1)$ such that $H_2(\rho) = 0$, that is,

$$[g_{11}(0)(\widetilde{g}_{22}(\rho) - \rho) - g_{21}(0)\widetilde{g}_{12}(\rho)]\phi_1(0) = [g_{22}(0)\widetilde{g}_{12}(\rho) - g_{12}(0)(\widetilde{g}_{22}(\rho) - \rho)]\phi_2(0). \tag{18}$$

If we know the the value of ρ, then the relation between $\phi_1(0)$ and $\phi_2(0)$ is obvious. Noting that $\widetilde{\phi}_1(s) > 0$ for any $s \in (0, 1)$, we see that ρ is a solution to $H_1(s) = 0$.

Case 3. If $f_0 < 0$, then $H_1(0) = f_0 < 0$. On the other hand,

$$\begin{aligned} H_1(-1) &= (1 + \widetilde{g}_{11}(-1))(1 + \widetilde{g}_{22}(-1)) - \widetilde{g}_{21}(-1)\widetilde{g}_{12}(-1) \\ &> (1 - \widetilde{g}_{11}(1))(1 - \widetilde{g}_{22}(1)) - \widetilde{g}_{21}(1)\widetilde{g}_{12}(1) \\ &= (1 - p_{11})(1 - p_{22}) - p_{21}p_{12} = 0. \end{aligned}$$

So there exists a $\rho \in (-1, 0)$ such that $H_1(\rho) = 0$, which in turn implies that $H_2(\rho) = 0$. That is, (18) also holds in this case.

5. Some special cases

In this section, we discuss a few special cases of model (1) which have been considered in the literature.

5.1. The compound binomial model

In the compound binomial model, the surplus of an insurance company at time t is defined by

$$U_t = u + t - \sum_{i=1}^{t} I_iX_i, \qquad t \in \mathbb{N}_+, \tag{19}$$

where the claim amounts X_i are independent and identically distributed (i.i.d.) positive integer-valued random variables with common probability function $f(k) = \mathbb{P}(X = k), k = 1, 2, \ldots$, and

I_i are i.i.d. Bernoulli random variables with mean $q \in (0,1)$. That is, in any time period, the probability of having a claim is q and the probability of no claim is $1-q=p$. The compound binomial model was first proposed by Gerber [16], and then extensively studied by Shiu [17], Willmot [18], Dickson [19], Cheng et al. [20], Tan and Yang [21], and so on.

For $i=1,2$, if we let

$$g_{i1}(k)=\begin{cases} p, & k=0,\\ 0, & k>0,\end{cases} \qquad g_{i2}(k)=\begin{cases} 0 \quad , & k=0,\\ (1-p)f(k), & k>0,\end{cases}$$

then our model reduces to the compound binomial model. Let $\phi(u)=\phi_1(u)=\phi_2(u)$. In this case, we have $f_0=0$, and it follows from (10) that

$$\phi(u)=\frac{1}{p}\Big[\phi(u-1)-q\sum_{n=0}^{u-1}\phi(n)f(u-n)\Big], u=1,2,\dots. \tag{20}$$

By (15), we obtain $\phi(0)=(1-\mu q)/p$, where $\mu=\sum_{k=1}^{\infty}kf(k)$. This result is the same as that in Shiu [17] and Willmot [18].

5.2. The compound Markov binomial model

An extension of the compound binomial model is the compound Markov binomial model, in which $\{I_k, k\in\mathbb{N}\}$ in (19) is a two-state Markovian process with a transition probability matrix $\mathbf{P}=(p_{ij})_{i,j\in\{1,2\}}$, where $p_{ij}=\mathbb{P}(I_{k+1}=j-1|I_k=i-1)$ for $i,j\in\{1,2\}$ and $k\in\mathbb{N}$. Some important references on the compound Markov binomial model include Cossette et al. [22, 23] and Yuen and Guo [24].

For $i=1,2$, if we let

$$g_{i1}(k)=\begin{cases} p_{i1}, & k=0,\\ 0\ , & k>0,\end{cases} \qquad g_{i2}(k)=\begin{cases} 0 \quad , & k=0,\\ p_{i2}f(k), & k>0,\end{cases}$$

then one can see that the compound Markov binomial model is a special case of our model. In this case, we have $f_0=0$. Again it follows from (10) that

$$\phi_2(u)=-\frac{1}{f_1}\Big[\phi_2(u-1)+\sum_{n=0}^{u-1}\phi_2(n)[(p_{11}-p_{21})f(u+1-n)-p_{22}f(u-n)]\Big],\ u=1,2,\cdots. \tag{21}$$

By (17), we obtain

$$\psi_2(0)=\frac{(p_{11}-p_{21})(1-f(1))+p_{21}\psi_1(0)}{p_{11}-(p_{11}-p_{21})f(1)},$$

which is the same as (7) of Cossette et al. [23]. Besides, by mathematical induction, one can show that (21) is equivalent to (9) of Cossette et al. [23].

5.3. The compound binomial model with time-correlated claims

Another extension of the compound binomial model is the compound binomial model with time-correlated claims studied by Yuen and Guo [25] and Xiao and Guo [26]. It is assumed that every main claim produces a by-claim but the occurrence of the by-claim may be delayed. In any time period, the probability of having a main claim is $q, 0 < q < 1$, and thus the probability of no main claim is $p = 1 - q$. The by-claim and its associated main claim may occur simultaneously with probability θ, or the occurrence of the by-claim may be delayed to the next time period with probability $1 - \theta$. The main claim amounts $X_1, X_2, \ldots,$ are i.i.d. with mean μ_X; and the by-claim amounts $Y_0, Y_1, Y_2, \ldots,$ are i.i.d. with mean μ_Y; and all the X_i's and Y_i's are independent.

Define the environmental Markov chain $(J_n; n \in \mathbb{N})$ in the following way: $\{J_t = 1\}$ implies that the by-claim Y_{t-1} in time period $t-1$ should not be delayed if it exists, while $\{J_t = 2\}$ means that there is a by-claim Y_{t-1} occurring in time period t. Then $g_{ij}(k)$ is given by

$$g_{1j}(k) = \begin{cases} p & , \; j=1, \; k=0, \\ q\theta\mathbb{P}(X_1 + Y_1 = k), & j=1, \; k>0, \\ 0 & , \; j=2, \; k=0, \\ q(1-\theta)\mathbb{P}(X_1 = k), & j=2, \; k>0, \end{cases}$$

$$g_{2j}(k) = \begin{cases} 0 & , \; j=1,2, \; k=0, \\ p\mathbb{P}(Y_0 = k) + q\theta\mathbb{P}(X_1 + Y_1 + Y_0 = k), & j=1, \; k>0, \\ q(1-\theta)\mathbb{P}(X_1 + Y_0 = k) & , \; j=2, \; k>0. \end{cases}$$

As a result, we have $f_0 = 0$. Using (15), we obtain

$$\psi_1(0) = \frac{q[\mu_X + \mu_Y - 1 - p(1-\theta)]}{p(p+q\theta)},$$

which is the same as (15) of Xiao and Guo [26]. Also $\phi_2(0)$ and the recursive formulae for computing $\phi_i(u)$ can be obtained using (17) and (10) respectively.

5.4. The compound Markov binomial model with time-correlated claims

A more general extension of the compound binomial model is the compound Markov binomial model with time-correlated claims. Specifically, suppose that $\{I_k, k \in \mathbb{N}\}$ in (19) is a two-state Markovian process with a transition probability matrix $\mathbf{P} = (p_{ij})_{i,j \in \{1,2\}}$, where $p_{ij} = \mathbb{P}(I_{k+1} = j-1 | I_k = i-1)$ for $i, j \in \{1, 2\}$ and $k \in \mathbb{N}$. The two types of claims, namely the main claim and by-claim, are defined in Section 5.3.

In this case, the environmental Markov chain $(J_n; n \in \mathbb{N})$ is defined by $J_n = I_n + 1, \forall n \in \mathbb{N}$. Then $g_{ij}(k)$ is given by

$$g_{1j}(k) = \begin{cases} p_{11} & , \; j=1, \; k=0, \\ 0 & , \; j=1, \; k>0, \\ 0 & , \; j=2, \; k=0, \\ p_{12}[(1-\theta)\mathbb{P}(X_1 = k) + \theta\mathbb{P}(X_1 + Y_1 = k)], & j=2, \; k>0, \end{cases}$$

$$g_{2j}(k) = \begin{cases} p_{21}\theta & , \; j=1, \; k=0, \\ p_{21}(1-\theta)\mathbb{P}(Y_1 = k) & , \; j=1, \; k>0, \\ 0 & , \; j=2, \; k=0, \\ p_{22}[(1-\theta)^2\mathbb{P}(Y_0 + X_1 = k) + \theta^2\mathbb{P}(X_1 + Y_1 = k) & \\ \quad + \theta(1-\theta)(\mathbb{P}(X_1 = k) + \mathbb{P}(X_1 + Y_1 + Y_0 = k))], & j=2, \; k>0. \end{cases}$$

In this case, $f_0 = 0$. By (15), we obtain

$$\phi_1(0) = \frac{p_{12} + p_{21} - p_{12}(\mu_X + \mu_Y)}{p_{21}(p_{11} + p_{12}\theta)}.$$

Using (17) and (10), one can derive $\phi_2(0)$ and the recursive formulae for computing $\phi_i(u)$, respectively.

6. Numerical examples

In this section, we provide three numerical examples to illustrate the application of our results. These examples cover each of the three cases in Section 4.2. We start with the example considered by Reinhard and Snoussi [1, 2].

Example 1 (Case 1). The distribution of claims $g_{ij}(k)$ is given in Table 1.

[Table 1.]

By direct calculation, we have

$$p_{12} = \frac{1}{8}, \quad p_{21} = \frac{2}{3}, \quad \mu_1 = \frac{1}{2}, \quad \mu_2 = 2, \quad f_0 = 0, \quad f_1 = -\frac{25}{48}.$$

Then it follows from (15) and (17) that $\phi_1(0) = 1/2$ and $\phi_2(0) = 0$. Furthermore,

$$f_2 = \frac{5}{6}, \quad f_3 = -\frac{5}{16}, \quad f_k = 0, \ k \geq 4,$$
$$h_2^{(1)} = \xi_2^{(1)}\phi_1(0) = \frac{5}{48}\phi_1(0), \quad h_k^{(1)} = 0, \ k \geq 3.$$

So we have

$$\phi_1(1) = \frac{1}{f_1}(h_2^{(1)} - f_2\phi_1(0)) = \frac{7}{10}.$$

Using (6), we obtain

$$f_1\phi_1(k) = -f_2\phi_1(k-1) - f_3\phi_1(k-2) = (f_1 + f_3)\phi_1(k-1) - f_3\phi_1(k-2), \quad k \geq 2,$$

that is,

$$\phi_1(k) - \phi_1(k-1) = \frac{f_3}{f_1}[\phi_1(k-1) - \phi_1(k-2)], \quad k \geq 2.$$

Therefore,

$$\phi_1(k) - \phi_1(k-1) = [\phi_1(1) - \phi_1(0)]\Big(\frac{f_3}{f_1}\Big)^{k-1} = \frac{1}{5}\Big(\frac{3}{5}\Big)^{k-1}, \quad k \geq 1,$$

which yields

$$\phi_1(k) = 1 - \frac{1}{2}\Big(\frac{3}{5}\Big)^{k}, \quad k \geq 1.$$

Similarly we get

$$\phi_2(k) = 1 - \frac{7}{10}\Big(\frac{3}{5}\Big)^{k-1}, \quad k \geq 1.$$

One can see that the results obtained above are the same as those in Reinhard and Snoussi [1, 2].

Example 2 (Case 2). The distribution of claims $g_{ij}(k)$ is given in Table 2.

[Table 2.]

By direct calculation, we have

$$p_{11} = \frac{5}{8}, \quad p_{12} = \frac{3}{8}, \quad p_{21} = \frac{1}{3}, \quad p_{22} = \frac{2}{3},$$
$$\mu_1 = \mu_{11} + \mu_{12} = \frac{1}{2} + \frac{3}{8} = \frac{7}{8}, \quad \mu_2 = \mu_{21} + \mu_{22} = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}.$$

Then by (15), we obtain

$$\frac{1}{8}\phi_1(0) + \frac{11}{48}\phi_2(0) = \frac{5}{48}. \tag{22}$$

Since $f_0 = 3/16 > 0$, this example corresponds to Case 2 of Section 4.2. Besides,

$$\widetilde{g}_{11}(s) = \frac{1}{8}(3 + s + s^3), \quad \widetilde{g}_{12}(s) = \frac{1}{8}(1 + s + s^2),$$
$$\widetilde{g}_{21}(s) = \frac{1}{12}(3s + s^3), \quad \widetilde{g}_{22}(s) = \frac{1}{6}(3 + s^2),$$
$$H_1(s) = \frac{1}{96}(1 - s)(-s^4 + 12s^3 + 24s^2 - 63s + 18).$$

Using Matlab, we get $\rho = 0.335626627250400$, which is the unique solution to $H_1(s) = 0$ on $(0, 1)$. Combining (18) and (22) with $\rho = 0.335626627250400$, we obtain

$$\phi_1(0) = 0.291173297926802, \quad \phi_2(0) = 0.295723655676290.$$

Having obtained the initial values, we can use (10) to calculate the values of $\phi_i(u),\ u \in \mathbb{N}$. Some of the values are given in Table 3.

[Table 3.]

Remark 3. *The implementation of (10) requires a high degree of accuracy. If the figures in Table 3 are accurate to four decimal places, it will lead to some perverse results. For example, if the values of ρ, $\phi_1(0)$ and $\phi_2(0)$ are given by*

$$\rho = 0.3356, \quad \phi_1(0) = 0.2912, \quad \phi_2(0) = 0.2957,$$

we obtain

$$\phi_1(8) = 1.1225, \quad \phi_1(9) = 1.5661, \ldots,$$

which are quite absurd.

Finally, we deal with an example which corresponds to Case 3 of Section 4.2.

Example 3 (Case 3). In this example, $g_{ij}(k)$ is given by $g_{ij}(k) = p_{ij} g_j(k)$ with

$$\mathbf{P} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}, \quad g_1(k) = \left(\frac{1}{2}\right)^{k+1}, \quad g_2(k) = \frac{2}{3}\left(\frac{1}{3}\right)^k, \ k \in \mathbb{N}.$$

By direct calculation, we have $f_0 = -5/36 < 0$,

$$\mu_1 = \mu_{11} + \mu_{12} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}, \quad \mu_2 = \mu_{21} + \mu_{22} = \frac{3}{4} + \frac{1}{8} = \frac{7}{8}.$$

Then by (15), we obtain

$$\frac{3}{8}\phi_1(0) + \frac{4}{9}\phi_2(0) = \frac{1}{3}. \tag{23}$$

On the other hand,

$$\begin{aligned} \widetilde{g}_{11}(s) &= \sum_{k=0}^{\infty} s^k g_{11}(k) = \frac{1}{3(2-s)}, \quad \widetilde{g}_{12}(s) = \frac{4}{3(3-s)}, \\ \widetilde{g}_{21}(s) &= \frac{3}{4(2-s)}, \quad \widetilde{g}_{22}(s) = \frac{1}{2(3-s)}, \\ H_1(s) &= \frac{(s-1)(6s^3 - 24s^2 + 17s + 5)}{6(2-s)(3-s)}. \end{aligned}$$

Again, using Matlab, we get $\rho = -0.221212431707700$, which is the unique solution to $H_1(s) = 0$ on $(-1, 0)$. Combining (18) and (23) with $\rho = -0.221212431707700$, we obtain

$$\phi_1(0) = 0.420307913413719, \quad \phi_2(0) = 0.395365198057175.$$

Then the values of $\phi_i(u)$, $u \in \mathbb{N}$, can be calculated using (10). Some of these values are given in Table 4.

[Table 4.]

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